This Teaching Guide was collaboratively developed and reviewed by educators from public and private schools, colleges, and universities. We encourage teachers and other education stakeholders to email their feedback, comments, and recommendations to the Commission on Higher Education, K to 12 Transition Program Management Unit - Senior High School Support Team at k12@ched.gov.ph. We value your feedback and recommendations.
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Introduction

As the Commission supports DepEd’s implementation of Senior High School (SHS), it upholds the vision and mission of the K to 12 program, stated in Section 2 of Republic Act 10533, or the Enhanced Basic Education Act of 2013, that “every graduate of basic education be an empowered individual, through a program rooted on...the competence to engage in work and be productive, the ability to coexist in fruitful harmony with local and global communities, the capability to engage in creative and critical thinking, and the capacity and willingness to transform others and oneself.”

To accomplish this, the Commission partnered with the Philippine Normal University (PNU), the National Center for Teacher Education, to develop Teaching Guides for Courses of SHS. Together with PNU, this Teaching Guide was studied and reviewed by education and pedagogy experts, and was enhanced with appropriate methodologies and strategies.

Furthermore, the Commission believes that teachers are the most important partners in attaining this goal. Incorporated in this Teaching Guide is a framework that will guide them in creating lessons and assessment tools, support them in facilitating activities and questions, and assist them towards deeper content areas and competencies. Thus, the introduction of the SHS for SHS Framework.

The SHS for SHS Framework

The SHS for SHS Framework, which stands for “Saysay-Husay-Sarili for Senior High School,” is at the core of this book. The lessons, which combine high-quality content with flexible elements to accommodate diversity of teachers and environments, promote these three fundamental concepts:

**SAYSAY: MEANING**
*Why is this important?*

Through this Teaching Guide, teachers will be able to facilitate an understanding of the value of the lessons, for each learner to fully engage in the content on both the cognitive and affective levels.

**HUSAY: MASTERY**
*How will I deeply understand this?*

Given that developing mastery goes beyond memorization, teachers should also aim for deep understanding of the subject matter where they lead learners to analyze and synthesize knowledge.

**SARILI: OWNERSHIP**
*What can I do with this?*

When teachers empower learners to take ownership of their learning, they develop independence and self-direction, learning about both the subject matter and themselves.
K to 12 BASIC EDUCATION CURRICULUM
SENIOR HIGH SCHOOL – SCIENCE, TECHNOLOGY, ENGINEERING AND MATHEMATICS (STEM) SPECIALIZED SUBJECT

Grade: 11
Core Subject Title: Pre-Calculus  
Semester: First Semester
No. of Hours/ Semester: 80 hours/ semester
Pre-requisite (if needed):

Subject Description: At the end of the course, the students must be able to apply concepts and solve problems involving conic sections, systems of nonlinear equations, series and mathematical induction, circular and trigonometric functions, trigonometric identities, and polar coordinate system.

<table>
<thead>
<tr>
<th>CONTENT</th>
<th>CONTENT STANDARDS</th>
<th>PERFORMANCE STANDARDS</th>
<th>LEARNING COMPETENCIES</th>
<th>CODE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytic Geometry</td>
<td>The learners demonstrate an understanding of... key concepts of conic sections and systems of nonlinear equations</td>
<td>The learners shall be able to... model situations appropriately and solve problems accurately using conic sections and systems of nonlinear equations</td>
<td>The learners... 1. illustrate the different types of conic sections: parabola, ellipse, circle, hyperbola, and degenerate cases.*** 2. define a circle. 3. determine the standard form of equation of a circle 4. graph a circle in a rectangular coordinate system 5. define a parabola 6. determine the standard form of equation of a parabola 7. graph a parabola in a rectangular coordinate system 8. define an ellipse 9. determine the standard form of equation of an ellipse 10. graph an ellipse in a rectangular coordinate system 11. define a hyperbola 12. determine the standard form of equation of a hyperbola</td>
<td>STEM_PC11AG-Ia-1 STEM_PC11AG-Ia-2 STEM_PC11AG-Ia-3 STEM_PC11AG-Ia-4 STEM_PC11AG-Ia-5 STEM_PC11AG-Ib-1 STEM_PC11AG-Ib-2 STEM_PC11AG-Ic-1 STEM_PC11AG-Ic-2 STEM_PC11AG-Ic-3 STEM_PC11AG-Id-1 STEM_PC11AG-Id-2</td>
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<td>CONTENT</td>
<td>CONTENT STANDARDS</td>
<td>PERFORMANCE STANDARDS</td>
<td>LEARNING COMPETENCIES</td>
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<tr>
<td>Series and Mathematical Induction</td>
<td>key concepts of series and mathematical induction and the Binomial Theorem.</td>
<td>keenly observe and investigate patterns, and formulate appropriate mathematical statements and prove them using mathematical induction and/or Binomial Theorem.</td>
<td>1. illustrate a series</td>
<td>STEM_PC11SMI-Ih-1</td>
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<tr>
<td></td>
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<td>2. differentiate a series from a sequence</td>
<td>STEM_PC11SMI-Ih-2</td>
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<td>3. use the sigma notation to represent a series</td>
<td>STEM_PC11SMI-Ih-3</td>
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<td>4. illustrate the Principle of Mathematical Induction</td>
<td>STEM_PC11SMI-Ih-4</td>
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<td>5. apply mathematical induction in proving identities</td>
<td>STEM_PC11SMI-Ih-i-1</td>
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<td>6. illustrate Pascal’s Triangle in the expansion of ((x + y)^n) for small positive integral values of (n)</td>
<td>STEM_PC11SMI-II-2</td>
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<td>7. prove the Binomial Theorem</td>
<td>STEM_PC11SMI-II-3</td>
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<td>8. determine any term of ((x + y)^n), where (n) is a positive integer, without expanding</td>
<td>STEM_PC11SMI-Ij-1</td>
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<td>9. solve problems using mathematical induction and the Binomial Theorem</td>
<td>STEM_PC11SMI-Ij-2</td>
</tr>
</tbody>
</table>

13. graph a hyperbola in a rectangular coordinate system  
14. recognize the equation and important characteristics of the different types of conic sections  
15. solves situational problems involving conic sections  
16. illustrate systems of nonlinear equations  
17. determine the solutions of systems of nonlinear equations using techniques such as substitution, elimination, and graphing***  
18. solve situational problems involving systems of nonlinear equations
<table>
<thead>
<tr>
<th>CONTENT</th>
<th>CONTENT STANDARDS</th>
<th>PERFORMANCE STANDARDS</th>
<th>LEARNING COMPETENCIES</th>
<th>CODE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trigonometry</td>
<td>key concepts of circular functions, trigonometric identities, inverse trigonometric functions, and the polar coordinate system</td>
<td>1. formulate and solve accurately situational problems involving circular functions</td>
<td>1. illustrate the unit circle and the relationship between the linear and angular measures of a central angle in a unit circle</td>
<td>STEM_PC11T-IIa-1</td>
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<tr>
<td></td>
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<td>2. convert degree measure to radian measure and vice versa</td>
<td>2.. STEM_PC11T-IIa-2</td>
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<td>3. illustrate angles in standard position and coterminal angles</td>
<td>3. STEM_PC11T-IIa-3</td>
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<td>4. illustrate the different circular functions</td>
<td>4. STEM_PC11T-IIb-1</td>
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<td>5. uses reference angles to find exact values of circular functions</td>
<td>5. STEM_PC11T-IIb-2</td>
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<td>6. determine the domain and range of the different circular functions</td>
<td>6. STEM_PC11T-IIc-1</td>
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<td>7. graph the six circular functions (a) amplitude, (b) period, and (c) phase shift</td>
<td>7. STEM_PC11T-IId-1</td>
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<td>8. solve problems involving circular functions</td>
<td>8. STEM_PC11T-IId-2</td>
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<td>9. determine whether an equation is an identity or a conditional equation</td>
<td>9. STEM_PC11T-IIe-1</td>
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<td></td>
<td>10. derive the fundamental trigonometric identities</td>
<td>10. STEM_PC11T-IIe-2</td>
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<td>11. derive trigonometric identities involving sum and difference of angles</td>
<td>11. STEM_PC11T-IIe-3</td>
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<td>12. derive the double and half-angle formulas</td>
<td>12. STEM_PC11T-IIf-1</td>
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<td>13. simplify trigonometric expressions</td>
<td>13. STEM_PC11T-IIf-2</td>
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<td>14. prove other trigonometric identities</td>
<td>14. STEM_PC11T-IIg-1</td>
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<tr>
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<td></td>
<td>15. solve situational problems involving trigonometric identities</td>
<td>15. STEM_PC11T-IIg-2</td>
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<td>16. illustrate the domain and range of the inverse trigonometric functions</td>
<td>16. STEM_PC11T-IIh-1</td>
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<td>17. evaluate an inverse trigonometric expression</td>
<td>17. STEM_PC11T-IIh-2</td>
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<td>18. solve trigonometric equations</td>
<td>18. STEM_PC11T-IIh-i-1</td>
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<td>19. solve situational problems involving inverse trigonometric functions and trigonometric equations</td>
<td>19. STEM_PC11T-IIii-2</td>
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<tr>
<td></td>
<td></td>
<td>20. locate points in polar coordinate system</td>
<td>20. STEM_PC11T-IIj-1</td>
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<td></td>
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<td>21. convert the coordinates of a point from rectangular to polar systems and vice versa</td>
<td>21. STEM_PC11T-IIj-2</td>
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<tr>
<td></td>
<td></td>
<td>22. solve situational problems involving polar coordinate system</td>
<td>22. STEM_PC11T-IIj-3</td>
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</tbody>
</table>

***Suggestion for ICT-enhanced lesson when available and where appropriate***
## Code Book Legend

**Sample:** STEM_PC11AG-Ia-1

<table>
<thead>
<tr>
<th>LEGEND</th>
<th>SAMPLE</th>
<th>DOMAIN/ COMPONENT</th>
<th>CODE</th>
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<td>First Entry</td>
<td>Learning Area and Strand/ Subject or Specialization</td>
<td>Science, Technology, Engineering and Mathematics Pre-Calculus</td>
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<td>Grade Level</td>
<td>Grade 11</td>
<td></td>
</tr>
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<td>Uppercase Letter/s</td>
<td>Domain/Content/ Component/ Topic</td>
<td>Analytic Geometry</td>
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<tr>
<td>Roman Numeral</td>
<td>Quarter</td>
<td>First Quarter</td>
<td>i</td>
</tr>
<tr>
<td><em>Zero if no specific quarter</em></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lowercase Letter/s</td>
<td>Week</td>
<td>Week one</td>
<td>a</td>
</tr>
<tr>
<td><em>Put a hyphen (-) in between letters to indicate more than a specific week</em></td>
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<tr>
<td>Arabic Number</td>
<td>Competency</td>
<td>illustrate the different types of conic sections: parabola, ellipse, circle, hyperbola, and degenerate cases</td>
<td>1</td>
</tr>
</tbody>
</table>

### Key Terms
- **AG:** Analytic Geometry
- **SMI:** Series and Mathematical Induction
- **T:** Trigonometry
About this Teaching Guide

The Precalculus course bridges Basic Mathematics and Calculus. This course completes the foundational knowledge on Algebra, Geometry, and Trigonometry of students who are planning to take courses in the STEM track. It provides them with conceptual understanding and computational skills that are crucial for Basic Calculus and future STEM courses.

Based on the Curriculum Guide for Precalculus of the Department of Education, the primary aim of this Teaching Guide is to give Math teachers adequate stand-alone material that can be used for each session of the Grade 11 Precalculus course.

The Guide is divided into three units: Analytic Geometry, Summation Notation and Mathematical Induction, and Trigonometry. Each unit is composed of lessons that bring together related learning competencies in the unit. Each lesson is further divided into sub-lessons that focus on one or two competencies for effective teaching and learning. Each sub-lesson is designed for a one-hour session, but the teachers have the option to extend the time allotment to one-and-a-half hours for some sub-lessons.

Each sub-lesson ends with a Seatwork / Homework, which consists of exercises related to the topic being discussed in the sub-lesson. As the title suggests, these exercises can be done in school (if time permits) or at home. Moreover, at the end of each lesson is a set of exercises (simply tagged as Exercises) that can be used for short quizzes and long exams. Answers, solutions, or hints to most items in Seatwork / Homework and Exercises are provided to guide the teachers as they solve them.

Some items in this Guide are marked with a star. A starred sub-lesson is optional and it is suggested that these be taken only if time permits. A starred example or exercise requires the use of a calculator.

To further guide the teachers, Teaching Notes are provided on the margins. These notes include simple recall of basic definitions and theorems, suggested teaching methods, alternative answers to some exercises, quick approaches and techniques in solving particular problems, and common errors committed by students.

We hope that Precalculus teachers will find this Teaching Guide helpful and convenient to use. We encourage the teachers to study this Guide carefully and solve the exercises themselves. Although great effort has been put to this Guide for technical correctness and precision, any mistake found and reported to the Team is a gain for other teachers. Thank you for your cooperation.
The Parts of the Teaching Guide

This Teaching Guide is mapped and aligned to the DepEd SHS Curriculum, designed to be highly usable for teachers. It contains classroom activities and pedagogical notes, and integrated with innovative pedagogies. All of these elements are presented in the following parts:

1. **INTRODUCTION**
   - Highlight key concepts and identify the essential questions
   - Show the big picture
   - Connect and/or review prerequisite knowledge
   - Clearly communicate learning competencies and objectives
   - Motivate through applications and connections to real-life

2. **MOTIVATION**
   - Give local examples and applications
   - Engage in a game or movement activity
   - Provide a hands-on/laboratory activity
   - Connect to a real-life problem

3. **INSTRUCTION/DELIVERY**
   - Give a demonstration/lecture/simulation/hands-on activity
   - Show step-by-step solutions to sample problems
   - Give applications of the theory
   - Connect to a real-life problem if applicable

4. **PRACTICE**
   - Provide easy-medium-hard questions
   - Give time for hands-on unguided classroom work and discovery
   - Use formative assessment to give feedback

5. **ENRICHMENT**
   - Provide additional examples and applications
   - Introduce extensions or generalisations of concepts
   - Engage in reflection questions
   - Encourage analysis through higher order thinking prompts
   - Allow pair/small group discussions
   - Summarize and synthesize the learnings

6. **EVALUATION**
   - Supply a diverse question bank for written work and exercises
   - Provide alternative formats for student work: written homework, journal, portfolio, group/individual projects, student-directed research project
On DepEd Functional Skills and CHED’s College Readiness Standards

As Higher Education Institutions (HEIs) welcome the graduates of the Senior High School program, it is of paramount importance to align Functional Skills set by DepEd with the College Readiness Standards stated by CHED.

The DepEd articulated a set of 21st century skills that should be embedded in the SHS curriculum across various subjects and tracks. These skills are desired outcomes that K to 12 graduates should possess in order to proceed to either higher education, employment, entrepreneurship, or middle-level skills development.

On the other hand, the Commission declared the College Readiness Standards that consist of the combination of knowledge, skills, and reflective thinking necessary to participate and succeed - without remediation - in entry-level undergraduate courses in college.

The alignment of both standards, shown below, is also presented in this Teaching Guide - prepares Senior High School graduates to the revised college curriculum which will initially be implemented by AY 2018-2019.

<table>
<thead>
<tr>
<th>College Readiness Standards Foundational Skills</th>
<th>DepEd Functional Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Produce all forms of texts (written, oral, visual, digital) based on:</strong></td>
<td>Visual and information literacies</td>
</tr>
<tr>
<td>1. Solid grounding on Philippine experience and culture;</td>
<td>Media literacy</td>
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<tr>
<td>2. An understanding of the self, community, and nation;</td>
<td>Critical thinking and problem solving skills</td>
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<td>3. Application of critical and creative thinking and doing processes;</td>
<td>Creativity</td>
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<tr>
<td>4. Competency in formulating ideas/arguments logically, scientifically, and creatively; and</td>
<td>Initiative and self-direction</td>
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<td>5. Clear appreciation of one’s responsibility as a citizen of a multicultural Philippines and a diverse world;</td>
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<tr>
<td><strong>Systematically apply knowledge, understanding, theory, and skills for the development of the self, local, and global communities using prior learning, inquiry, and experimentation</strong></td>
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<td>Global awareness</td>
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<td>Scientific and economic literacy</td>
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<td>Curiosity</td>
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<td>Critical thinking and problem solving skills</td>
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<td>Risk taking</td>
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<td>Flexibility and adaptability</td>
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<td>Initiative and self-direction</td>
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<tr>
<td><strong>Work comfortably with relevant technologies and develop adaptations and innovations for significant use in local and global communities;</strong></td>
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<td>Global awareness</td>
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<td>Media literacy</td>
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<td>Technological literacy</td>
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<td>Creativity</td>
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<td>Flexibility and adaptability</td>
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<td>Productivity and accountability</td>
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<tr>
<td><strong>Communicate with local and global communities with proficiency, orally, in writing, and through new technologies of communication;</strong></td>
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<td>Global awareness</td>
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<td>Multicultural literacy</td>
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<td>Collaboration and interpersonal skills</td>
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<td>Social and cross-cultural skills</td>
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<td>Leadership and responsibility</td>
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Stretching from Samar to Leyte with a total length of more than two kilometers, the San Juanico Bridge has served as one of the main thoroughfares of economic and social development in the country since its completion in 1973. Adding picturesque effect on the whole architecture, geometric structures are subtly built to serve other purposes. The arch-shaped support on the main span of the bridge helps maximize its strength to withstand mechanical resonance and aeroelastic flutter brought about by heavy vehicles and passing winds.
Lesson 1.1. Introduction to Conic Sections and Circles

Time Frame: 4 one-hour sessions

Learning Outcomes of the Lesson

At the end of the lesson, the student is able to:

(1) illustrate the different types of conic sections: parabola, ellipse, circle, hyperbola, and degenerate cases;

(2) define a circle;

(3) determine the standard form of equation of a circle;

(4) graph a circle in a rectangular coordinate system; and

(5) solve situational problems involving conic sections (circles).

Lesson Outline

(1) Introduction of the four conic sections, along with the degenerate conics

(2) Definition of a circle

(3) Derivation of the standard equation of a circle

(4) Graphing circles

(5) Solving situational problems involving circles

Introduction

We introduce the conic sections, a particular class of curves which sometimes appear in nature and which have applications in other fields. In this lesson, we discuss the first of their kind, circles. The other conic sections will be covered in the next lessons.

1.1.1. An Overview of Conic Sections

We introduce the conic sections (or conics), a particular class of curves which oftentimes appear in nature and which have applications in other fields. One of the first shapes we learned, a circle, is a conic. When you throw a ball, the trajectory it takes is a parabola. The orbit taken by each planet around the sun is an ellipse. Properties of hyperbolas have been used in the design of certain telescopes and navigation systems. We will discuss circles in this lesson, leaving parabolas, ellipses, and hyperbolas for subsequent lessons.

- Circle (Figure 1.1) - when the plane is horizontal
- Ellipse (Figure 1.1) - when the (tilted) plane intersects only one cone to form a bounded curve
• Parabola (Figure 1.2) - when the plane intersects only one cone to form an unbounded curve
• Hyperbola (Figure 1.3) - when the plane (not necessarily vertical) intersects both cones to form two unbounded curves (each called a branch of the hyperbola)

![Figure 1.1](image1.png)  ![Figure 1.2](image2.png)  ![Figure 1.3](image3.png)

We can draw these conic sections (also called conics) on a rectangular coordinate plane and find their equations. To be able to do this, we will present equivalent definitions of these conic sections in subsequent sections, and use these to find the equations.

There are other ways for a plane and the cones to intersect, to form what are referred to as degenerate conics: a point, one line, and two lines. See Figures 1.4, 1.5 and 1.6.

![Figure 1.4](image4.png)  ![Figure 1.5](image5.png)  ![Figure 1.6](image6.png)

1.1.2. Definition and Equation of a Circle

A circle may also be considered a special kind of ellipse (for the special case when the tilted plane is horizontal). For our purposes, we will distinguish between these two conics.
See Figure 1.7, with the point \( C(3,1) \) shown. From the figure, the distance of \( A(-2,1) \) from \( C \) is \( AC = 5 \). By the distance formula, the distance of \( B(6,5) \) from \( C \) is \( BC = \sqrt{(6-3)^2 + (5-1)^2} = 5 \). There are other points \( P \) such that \( PC = 5 \). The collection of all such points which are 5 units away from \( C \), forms a circle.

![Figure 1.7](image1.png)

Let \( C \) be a given point. The set of all points \( P \) having the same distance from \( C \) is called a circle. The point \( C \) is called the center of the circle, and the common distance its radius.

The term radius is both used to refer to a segment from the center \( C \) to a point \( P \) on the circle, and the length of this segment.

See Figure 1.8, where a circle is drawn. It has center \( C(h,k) \) and radius \( r > 0 \). A point \( P(x,y) \) is on the circle if and only if \( PC = r \). For any such point then, its coordinates should satisfy the following.

\[
PC = r \\
\sqrt{(x-h)^2 + (y-k)^2} = r \\
(x-h)^2 + (y-k)^2 = r^2
\]

This is the standard equation of the circle with center \( C(h,k) \) and radius \( r \). If the center is the origin, then \( h = 0 \) and \( k = 0 \). The standard equation is then \( x^2 + y^2 = r^2 \).

**Example 1.1.1.** In each item, give the standard equation of the circle satisfying the given conditions.

1. center at the origin, radius 4
2. center \((-4,3)\), radius \( \sqrt{7} \)
(3) circle in Figure 1.7
(4) circle $A$ in Figure 1.9
(5) circle $B$ in Figure 1.9
(6) center $(5, -6)$, tangent to the $y$-axis
(7) center $(5, -6)$, tangent to the $x$-axis
(8) has a diameter with endpoints $A(-1, 4)$ and $B(4, 2)$

Solution. (1) $x^2 + y^2 = 16$
(2) $(x + 4)^2 + (y - 3)^2 = 7$
(3) The center is $(3, 1)$ and the radius is 5, so the equation is $(x - 3)^2 + (y - 1)^2 = 25$.
(4) By inspection, the center is $(-2, -1)$ and the radius is 4. The equation is $(x + 2)^2 + (y + 1)^2 = 16$.
(5) Similarly by inspection, we have $(x - 3)^2 + (y - 2)^2 = 9$.
(6) The center is 5 units away from the $y$-axis, so the radius is $r = 5$ (you can make a sketch to see why). The equation is $(x - 5)^2 + (y + 6)^2 = 25$.
(7) Similarly, since the center is 6 units away from the $x$-axis, the equation is $(x - 5)^2 + (y + 6)^2 = 36$. 
(8) The center $C$ is the midpoint of $A$ and $B$: $C = \left(\frac{-1+4}{2}, \frac{4+2}{2}\right) = \left(\frac{3}{2}, 3\right)$. The radius is then $r = AC = \sqrt{\left(-1 - \frac{3}{2}\right)^2 + (4 - 3)^2} = \sqrt{\frac{29}{4}}$. The circle has equation \((x - \frac{3}{2})^2 + (y - 3)^2 = \frac{29}{4}\).

**Seatwork/Homework 1.1.2**

Find the standard equation of the circle being described in each item.

(1) With center at the origin, radius $\sqrt{11}$ Answer: $x^2 + y^2 = 11$

(2) With center $(−6, 7)$, tangent to the $y$-axis Answer: $(x + 6)^2 + (y - 7)^2 = 36$

(3) Has a diameter with endpoints $A(-3, 2)$ and $B(7, 4)$ Answer: $(x - 2)^2 + (y - 3)^2 = 26$

**1.1.3. More Properties of Circles**

After expanding, the standard equation

\[
\left(x - \frac{3}{2}\right)^2 + (y - 3)^2 = \frac{29}{4}
\]

can be rewritten as

\[
x^2 + y^2 - 3x - 6y - 5 = 0,
\]

an equation of the circle in *general form*.

If the equation of a circle is given in the general form

\[
Ax^2 + Ay^2 + Cx + Dy + E = 0, \quad A \neq 0,
\]

or

\[
x^2 + y^2 + Cx + Dy + E = 0,
\]

we can determine the standard form by completing the square in both variables.

Completing the square in an expression like $x^2 + 14x$ means determining the term to be added that will produce a perfect polynomial square. Since the coefficient of $x^2$ is already 1, we take half the coefficient of $x$ and square it, and we get 49. Indeed, $x^2 + 14x + 49 = (x + 7)^2$ is a perfect square. To complete the square in, say, $3x^2 + 18x$, we factor the coefficient of $x^2$ from the expression: $3(x^2 + 6x)$, then add 9 inside. When completing a square in an equation, any extra term introduced on one side should also be added to the other side.

**Example 1.1.2.** Identify the center and radius of the circle with the given equation in each item. Sketch its graph, and indicate the center.

(1) $x^2 + y^2 - 6x = 7$
(2) \( x^2 + y^2 - 14x + 2y = -14 \)
(3) \( 16x^2 + 16y^2 + 96x - 40y = 315 \)

**Solution.** The first step is to rewrite each equation in standard form by completing the square in \( x \) and in \( y \). From the standard equation, we can determine the center and radius.

(1)

\[
\begin{align*}
\quad x^2 - 6x + y^2 &= 7 \\
\quad x^2 - 6x + 9 + y^2 &= 7 + 9 \\
\quad (x - 3)^2 + y^2 &= 16
\end{align*}
\]

Center \((3, 0), r = 4\), Figure 1.10

(2)

\[
\begin{align*}
\quad x^2 - 14x + y^2 + 2y &= -14 \\
\quad x^2 - 14x + 49 + y^2 + 2y + 1 &= -14 + 49 + 1 \\
\quad (x - 7)^2 + (y + 1)^2 &= 36
\end{align*}
\]

Center \((7, -1), r = 6\), Figure 1.11

(3)

\[
\begin{align*}
\quad 16x^2 + 96x + 16y^2 - 40y &= 315 \\
\quad 16(x^2 + 6x) + 16 \left( y^2 - \frac{5}{2}y \right) &= 315 \\
\quad 16(x^2 + 6x + 9) + 16 \left( y^2 - \frac{5}{2}y + \frac{25}{16} \right) &= 315 + 16(9) + 16 \left( \frac{25}{16} \right) \\
\quad 16(x + 3)^2 + 16 \left( y - \frac{5}{4} \right)^2 &= 484 \\
\quad (x + 3)^2 + \left( y - \frac{5}{4} \right)^2 &= \frac{484}{16} = \frac{121}{4} = \left( \frac{11}{2} \right)^2
\end{align*}
\]

Center \((-3, \frac{5}{4}), r = 5.5\), Figure 1.12.

**Teaching Notes**
A common mistake committed by students is to add 9 and \( \frac{25}{16} \) only. They often forget the multiplier outside the parenthesis.
In the standard equation \((x - h)^2 + (y - k)^2 = r^2\), both the two squared terms on the left side have coefficient 1. This is the reason why in the preceding example, we divided by 16 at the last equation.

**Seatwork/Homework 1.1.3**

Identify the center and radius of the circle with the given equation in each item. Sketch its graph, and indicate the center.

(1) \(x^2 + y^2 - 5x + 4y = 46\)

Answer: center \((\frac{5}{2}, -2)\), radius \(\frac{15}{2} = 7.5\), Figure 1.13

(2) \(4x^2 + 4y^2 + 40x - 32y = 5\)

Answer: center \((-5, 4)\), radius \(\frac{13}{2} = 6.5\), Figure 1.14

![Figure 1.13](image1.png) ![Figure 1.14](image2.png)

**1.1.4. Situational Problems Involving Circles**

We now consider some situational problems involving circles.

**Example 1.1.3.** A street with two lanes, each 10 ft wide, goes through a semicircular tunnel with radius 12 ft. How high is the tunnel at the edge of each lane? Round off to 2 decimal places.

![Diagram](image3.png)
Solution. We draw a coordinate system with origin at the middle of the highway, as shown. Because of the given radius, the tunnel’s boundary is on the circle $x^2 + y^2 = 12^2$. Point $P$ is the point on the arc just above the edge of a lane, so its $x$-coordinate is 10. We need its $y$-coordinate. We then solve $10^2 + y^2 = 12^2$ for $y > 0$, giving us $y = 2\sqrt{11} \approx 6.63$ ft.

Example 1.1.4. A piece of a broken plate was dug up in an archaeological site. It was put on top of a grid, as shown in Figure 1.15, with the arc of the plate passing through $A(-7, 0)$, $B(1, 4)$ and $C(7, 2)$. Find its center, and the standard equation of the circle describing the boundary of the plate.

Solution. We first determine the center. It is the intersection of the perpendicular
bisectors of $AB$ and $BC$ (see Figure 1.16). Recall that, in a circle, the perpendicular bisector of any chord passes through the center. Since the midpoint $M$ of $AB$ is \((-\frac{7+1}{2}, \frac{0+4}{2}) = (-3, 2)\), and $m_{AB} = \frac{4-0}{1+7} = \frac{1}{2}$, the perpendicular bisector of $AB$ has equation $y - 2 = -2(x + 3)$, or equivalently, $y = -2x - 4$.

Since the midpoint $N$ of $BC$ is \(\left(\frac{1+7}{2}, \frac{4+2}{2}\right) = (4, 3)\), and $m_{BC} = \frac{2-4}{7-1} = -\frac{1}{3}$, the perpendicular bisector of $BC$ has equation $y - 3 = 3(x - 4)$, or equivalently, $y = 3x - 9$.

The intersection of the two lines $y = 2x - 4$ and $y = 3x - 9$ is $(1, -6)$ (by solving a system of linear equations). We can take the radius as the distance of this point from any of $A$, $B$ or $C$ (it’s most convenient to use $B$ in this case). We then get $r = 10$. The standard equation is thus $(x - 1)^2 + (y + 6)^2 = 100$.

\[ \quad \]

**Seatwork/Homework 1.1.4**

*1. A single-lane street 10 ft wide goes through a semicircular tunnel with radius 9 ft. How high is the tunnel at the edge of each lane? Round off to 2 decimal places. Answer: 7.48 ft

2. An archeologist found the remains of an ancient wheel, which she then placed on a grid. If an arc of the wheel passes through $A(-7, 0)$, $B(-3, 4)$ and $C(7, 0)$, locate the center of the wheel, and the standard equation of the circle defining its boundary. Answer: $(0, -3)$, $x^2 + (y + 3)^2 = 58$

**Exercises 1.1**

1. Identify the center and radius of the circle with the given equation in each item. Sketch its graph, and indicate the center.

   \[
   \begin{align*}
   (a) \quad & x^2 + y^2 = 49 \quad \text{Answer: center } (0, 0), \quad r = 7 \\
   (b) \quad & 4x^2 + 4y^2 = 25 \quad \text{Answer: center } (0, 0), \quad r = \frac{5}{\sqrt{2}} \\
   (c) \quad & (x - \frac{7}{4})^2 + (y + \frac{3}{4})^2 = \frac{169}{16} \quad \text{Answer: center } \left(\frac{7}{4}, -\frac{3}{4}\right), \quad r = \frac{13}{4} \\
   (d) \quad & x^2 + y^2 - 12x - 10y = -12 \quad \text{Answer: center } (6, 5), \quad r = 7 \\
   (e) \quad & x^2 + y^2 + 8x - 9y = 6 \quad \text{Answer: center } (-4, 4.5), \quad r = \frac{13}{2} \\
   (f) \quad & x^2 + y^2 + 10x + 12y = -12 \quad \text{Answer: center } (-5, -6), \quad r = 7 \\
   (g) \quad & 2x^2 + 2y^2 - 14x + 18y = 7 \quad \text{Answer: center } (3.5, -4.5), \quad r = 6 \\
   (h) \quad & 4x^2 + 4y^2 - 20x + 40y = -5 \quad \text{Answer: center } (2.5, -5), \quad r = \sqrt{30} \\
   (i) \quad & 9x^2 + 9y^2 + 42x + 84y + 65 = 0 \quad \text{Answer: center } \left(-\frac{7}{3}, -\frac{14}{3}\right), \quad r = 2\sqrt{5} \\
   (j) \quad & 2x^2 + 2y^2 + 10x = 2y + 7 \quad \text{Answer: center } \left(-\frac{5}{2}, \frac{1}{2}\right), \quad r = \sqrt{10}
   \end{align*}
\]
2. Find the standard equation of the circle which satisfies the given conditions.

(a) center at the origin, radius $2\sqrt{2}$
   Answer: $x^2 + y^2 = 8$

(b) center at $(15, -20)$, radius 9
   Answer: $(x - 15)^2 + (y + 20)^2 = 81$

(c) center at $(5, 6)$, through $(9, 4)$
   Answer: $(x - 5)^2 + (y - 6)^2 = 20$

   Solution. The radius is the distance from the center to $(9, 4)$:
   
   $$\sqrt{(5 - 9)^2 + (6 - 4)^2} = \sqrt{20}.$$

(d) center at $(-2, 3)$, tangent to the $x$-axis
   Answer: $(x + 2)^2 + (y - 3)^2 = 9$

(e) center at $(-2, 3)$, tangent to the $y$-axis
   Answer: $(x + 2)^2 + (y - 3)^2 = 4$
(f) center at \((-2, 3)\), tangent to the line \(y = 8\)

Answer: \((x + 2)^2 + (y - 3)^2 = 25\)

Solution. We need to determine the radius. This is best done by sketching the center and line, to see that the center \((-2, 3)\) is 5 units away from the nearest point on the line, \((-2, 8)\) (which is the point of tangency).

(g) center at \((-2, 3)\), tangent to the line \(x = -10\)

Answer: \((x + 2)^2 + (y - 3)^2 = 64\)

(h) center in the third quadrant, tangent to both the \(x\)-axis and \(y\)-axis, radius 7

Answer: \((x + 7)^2 + (y + 7)^2 = 49\)

(i) a diameter with endpoints \((-9, 2)\) and \((15, 12)\)

Answer: \((x - 3)^2 + (y - 7)^2 = 169\)

(j) concentric with \(x^2 + y^2 + 2x - 4y = 5\), radius is 7

Answer: \((x + 1)^2 + (y - 2)^2 = 49\)

Solution. Two circles are said to be concentric if they have the same center. The standard equation of the given circle is \((x + 1)^2 + (y - 2)^2 = 10\). Thus, the circle we’re looking for has center \((-1, 2)\) and radius 7.

(k) concentric with \(x^2 + y^2 - 8x - 10y = -16\) and 4 times the area

Answer: \((x - 4)^2 + (y - 5)^2 = 100\)

Solution. The given circle has standard equation

\[(x - 4)^2 + (y - 5)^2 = 5^2.\]

Its radius is 5, so its area is \(25\pi\) sq. units. The circle we are looking for should have area \(100\pi\) sq. units, so its radius is 10.

(l) concentric with \(x^2 + y^2 - 10x - 6y = -2\), same radius as \(x^2 + y^2 - 14x + 6y = -33\)

Answer: \((x - 5)^2 + (y - 3)^2 = 25\)

(m) center at \(C(3, 4)\), tangent to the line \(y = \frac{1}{3}x - \frac{1}{3}\)

Answer: \((x - 3)^2 + (y - 4)^2 = 10\)

Solution. (A sketch will greatly help in understanding the argument.)

If \(P\) is the point of tangency, then line \(CP\) is perpendicular to the given tangent line. Since the tangent line has slope \(\frac{1}{3}\), line \(CP\) has slope \(-3\). Because it passes through \(C\), line \(CP\) has equation \(y - 4 = -3(x - 3)\), or \(y = -3x + 13\). Solving the system \{\(y = \frac{1}{3}x - \frac{1}{3}\), \(y = -3x + 13\)\} yields \(x = 4\) and \(y = 1\), the coordinates of \(P\). The radius is then \(CP = \sqrt{(3 - 4)^2 + (4 - 1)^2} = \sqrt{10}\).
(n) center at (−4, 3), tangent to the line \( y = -4x - 30 \)

Answer: \((x + 4)^2 + (y - 3)^2 = 17\)

**Solution.** (Similar to the previous problem) Let \( P \) be the point of
tangency, so line \( CP \) is perpendicular to the tangent line. The tangent
line has slope \(-4\), so line \( CP \) has slope \( \frac{1}{4} \). Line \( CP \) passes through \( C \),
so it has equation \( y - 3 = \frac{1}{4}(x + 4) \), or \( y = \frac{1}{4}x + 4 \). Solving the system
\( \{ y = -4x - 30, y = \frac{1}{4}x + 4 \} \) yields \( x = -8 \) and \( y = 2 \), the coordinates
of \( P \). The radius is then \( CP = \sqrt{(-4 + 8)^2 + (3 - 2)^2} = \sqrt{17} \).

*3. A seismological station is located at \((0, -3)\), 3 km away from a straight
shoreline where the \( x \)-axis runs through. The epicenter of an earthquake
was determined to be 6 km away from the station.

(a) Find the equation of the curve that contains the possible location of
the epicenter.

Answer: \( x^2 + (y + 3)^2 = 6^2 \)

(b) If furthermore, the epicenter was determined to be 2 km away from
the shore, find its possible coordinates (rounded off to two decimal
places).

Answer: \((\pm 3.32, 2)\)

**Solution.** Since the epicenter is 6 units away from \((0, -3)\), it could be any
of the points of a circle with center \((0, -3)\) and radius 6. The equation is
then \( x^2 + (y + 3)^2 = 6^2 \). Next, we solve this equation for \( x \) if \( y = 2 \), and we
get \( x^2 = 6^2 - (2 + 3)^2 = 11 \), and so \( x = \pm \sqrt{11} \approx \pm 3.32 \).

4. A ferris wheel is elevated 1 m above ground. When a car reaches the highest
point on the ferris wheel, its altitude from ground level is 31 m. How far
away from the center, horizontally, is the car when it is at an altitude of
25 m?

**Solution.** The ferris wheel, as shown
below, is drawn 1 unit above the \( x \)-
axis (ground level), center on the \( y \)-
axis, and highest point at \( y = 31 \).
The diameter is thus 30, and the ra-
dius 15. We locate the center at
\((0, 16)\), and write the equation of the
circle as \( x^2 + (y - 16)^2 = 15^2 \).

If \( y = 25 \), we have \( x^2 + (25 - 16)^2 = 15^2 \), so \( x^2 = 15^2 - 9^2 = 144 \), and
\( x = \pm 12 \). (Clearly, there are two
points on the ferris wheel at an altitude of 25 m.) Thus, the car is 12 m
away horizontally from the center.

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5. A window is to be constructed as shown, with its upper boundary the arc of a circle having radius 4 ft and center at the midpoint of base $AD$. If the vertical side is to be $\frac{3}{4}$ as long as the base, find the dimensions (vertical side and base) of this window. Round off your final answer to 1 decimal place.

Answer: base 4.44 ft, side 3.33 ft

Solution. We put two lines corresponding to the $x$-axis and $y$-axis, as shown, with the origin coinciding with the midpoint of the window’s base. This origin is the center of the circle containing the arc. The equation of the circle is then $x^2 + y^2 = 16$. Let $n$ be length of the base $AD$, so the side $AD$ has length $\frac{3}{4}n$. Point $B$ then has coordinates $\left(\frac{n}{2}, \frac{3n}{4}\right)$. Therefore, $\left(\frac{n}{2}\right)^2 + \left(\frac{3n}{4}\right)^2 = 16$. Solving this for $n > 0$ yields $n = \frac{16}{\sqrt{13}}$. The base is then $n \approx 4.44$ ft and the side $\frac{3}{4}n \approx 3.33$ ft.

Lesson 1.2. Parabolas

Time Frame: 3 one-hour sessions

Learning Outcomes of the Lesson

At the end of the lesson, the student is able to:

(1) define a parabola;
(2) determine the standard form of equation of a parabola;
(3) graph a parabola in a rectangular coordinate system; and
(4) solve situational problems involving conic sections (parabolas).
Lesson Outline

(1) Definition of a parabola
(2) Derivation of the standard equation of a parabola
(3) Graphing parabolas
(4) Solving situational problems involving parabolas

Introduction

A parabola is one of the conic sections. We have already seen parabolas which open upward or downward, as graphs of quadratic functions. Here, we will see parabolas opening to the left or right. Applications of parabolas are presented at the end.

1.2.1. Definition and Equation of a Parabola

Consider the point $F(0, 2)$ and the line $\ell$ having equation $y = -2$, as shown in Figure 1.17. What are the distances of $A(4, 2)$ from $F$ and from $\ell$? (The latter is taken as the distance of $A$ from $A_\ell$, the point on $\ell$ closest to $A$). How about the distances of $B(-8, 8)$ from $F$ and from $\ell$ (from $B_\ell$)?

$$AF = 4 \quad \text{and} \quad AA_\ell = 4$$

$$BF = \sqrt{(-8 - 0)^2 + (8 - 2)^2} = 10 \quad \text{and} \quad BB_\ell = 10$$

There are other points $P$ such that $PF = PP_\ell$ (where $P_\ell$ is the closest point on line $\ell$). The collection of all such points forms a shape called a parabola.

Let $F$ be a given point, and $\ell$ a given line not containing $F$. The set of all points $P$ such that its distances from $F$ and from $\ell$ are the same, is called a parabola. The point $F$ is its focus and the line $\ell$ its directrix.
Consider a parabola with focus \( F(0, c) \) and directrix \( \ell \) having equation \( x = -c \). See Figure 1.18. The focus and directrix are \( c \) units above and below, respectively, the origin. Let \( P(x, y) \) be a point on the parabola so \( PF = PP_\ell \), where \( P_\ell \) is the point on \( \ell \) closest to \( P \). The point \( P \) has to be on the same side of the directrix as the focus (if \( P \) was below, it would be closer to \( \ell \) than it is from \( F \)).

\[
PF = PP_\ell \\
\sqrt{x^2 + (y - c)^2} = y - (-c) = y + c \\
x^2 + y^2 - 2cy + c^2 = y^2 + 2cy + c^2 \\
x^2 = 4cy
\]

The vertex \( V \) is the point midway between the focus and the directrix. This equation, \( x^2 = 4cy \), is then the standard equation of a parabola opening upward with vertex \( V(0, 0) \).

Suppose the focus is \( F(0, -c) \) and the directrix is \( y = c \). In this case, a point \( P \) on the resulting parabola would be below the directrix (just like the focus). Instead of opening upward, it will open downward. Consequently, \( PF = \sqrt{x^2 + (y + c)^2} \) and \( PP_\ell = c - y \) (you may draw a version of Figure 1.18 for this case). Computations similar to the one done above will lead to the equation \( x^2 = -4cy \).

We collect here the features of the graph of a parabola with standard equation \( x^2 = 4cy \) or \( x^2 = -4cy \), where \( c > 0 \).

(1) vertex: origin \( V(0, 0) \)
- If the parabola opens upward, the vertex is the lowest point. If the parabola opens downward, the vertex is the highest point.

(2) directrix: the line \( y = -c \) or \( y = c \)
- The directrix is \( c \) units below or above the vertex.

(3) focus: \( F(0, c) \) or \( F(0, -c) \)
- The focus is \( c \) units above or below the vertex.
• Any point on the parabola has the same distance from the focus as it has from the directrix.

(4) **axis of symmetry**: $x = 0$ (the $y$-axis)

• This line divides the parabola into two parts which are mirror images of each other.

**Example 1.2.1.** Determine the focus and directrix of the parabola with the given equation. Sketch the graph, and indicate the focus, directrix, vertex, and axis of symmetry.

1. $x^2 = 12y$
2. $x^2 = -6y$

**Solution.**

1. The vertex is $V(0, 0)$ and the parabola opens upward. From $4c = 12$, $c = 3$. The focus, $c = 3$ units above the vertex, is $F(0, 3)$. The directrix, $3$ units below the vertex, is $y = -3$. The axis of symmetry is $x = 0$.

![Graph of $x^2 = 12y$]

2. The vertex is $V(0, 0)$ and the parabola opens downward. From $4c = 6$, $c = \frac{3}{2}$. The focus, $c = \frac{3}{2}$ units below the vertex, is $F(0, -\frac{3}{2})$. The directrix, $\frac{3}{2}$ units above the vertex, is $y = \frac{3}{2}$. The axis of symmetry is $x = 0$.

![Graph of $x^2 = -6y$]
**Example 1.2.2.** What is the standard equation of the parabola in Figure 1.17?

*Solution.* From the figure, we deduce that \( c = 2 \). The equation is thus \( x^2 = 8y \).

**Seatwork/Homework 1.2.1**

1. Give the focus and directrix of the parabola with equation \( x^2 = 10y \). Sketch the graph, and indicate the focus, directrix, vertex, and axis of symmetry. Answer: focus \((0, \frac{5}{2})\), directrix \( y = -\frac{5}{2} \)

![Graph of a parabola with focus at \((0, \frac{5}{2})\) and directrix \( y = -\frac{5}{2} \).](image)

2. Find the standard equation of the parabola with focus \( F(0, -3.5) \) and directrix \( y = 3.5 \). Answer: \( x^2 = -14y \)

**1.2.2. More Properties of Parabolas**

The parabolas we considered so far are “vertical” and have their vertices at the origin. Some parabolas open instead horizontally (to the left or right), and some have vertices not at the origin. Their standard equations and properties are given in the box. The corresponding computations are more involved, but are similar to the one above, and so are not shown anymore.

In all four cases below, we assume that \( c > 0 \). The vertex is \( V(h, k) \), and it lies between the focus \( F \) and the directrix \( \ell \). The focus \( F \) is \( c \) units away from the vertex \( V \), and the directrix is \( c \) units away from the vertex. Recall that, for any point on the parabola, its distance from the focus is the same as its distance from the directrix.
The following observations are worth noting.

- The equations are in terms of $x - h$ and $y - k$: the vertex coordinates are subtracted from the corresponding variable. Thus, replacing both $h$ and $k$ with 0 would yield the case where the vertex is the origin. For instance, this replacement applied to $(x - h)^2 = 4c(y - k)$ (parabola opening upward) would yield $x^2 = 4cy$, the first standard equation we encountered (parabola opening upward, vertex at the origin).

- If the $x$-part is squared, the parabola is “vertical”; if the $y$-part is squared, the parabola is “horizontal.” In a horizontal parabola, the focus is on the left or right of the vertex, and the directrix is vertical.

- If the coefficient of the linear (non-squared) part is positive, the parabola opens upward or to the right; if negative, downward or to the left.
**Example 1.2.3.** The figure shows the graph of parabola, with only its focus and vertex indicated. Find its standard equation. What is its directrix and its axis of symmetry?

![Graph of parabola with focus and vertex indicated]

**Solution.** The vertex is $V(5, -4)$ and the focus is $F(3, -4)$. From these, we deduce the following: $h = 5$, $k = -4$, $c = 2$ (the distance of the focus from the vertex). Since the parabola opens to the left, we use the template $(y - k)^2 = -4c(x - h)$. Our equation is

$$(y + 4)^2 = -8(x - 5).$$

Its directrix is $c = 2$ units to the right of $V$, which is $x = 7$. Its axis is the horizontal line through $V$: $y = -4$.

The standard equation $(y + 4)^2 = -8(x - 5)$ from the preceding example can be rewritten as $y^2 + 8x + 8y - 24 = 0$, an equation of the parabola in general form.

If the equation is given in the general form $Ax^2 + Cx + Dy + E = 0$ ($A$ and $C$ are nonzero) or $By^2 + Cx + Dy + E = 0$ ($B$ and $C$ are nonzero), we can determine the standard form by completing the square in both variables.

**Example 1.2.4.** Determine the vertex, focus, directrix, and axis of symmetry of the parabola with the given equation. Sketch the parabola, and include these points and lines.

(a) $y^2 - 5x + 12y = -16$  
(b) $5x^2 + 30x + 24y = 51$
Solution. (1) We complete the square on $y$, and move $x$ to the other side.

\[
y^2 + 12y = 5x - 16
\]

\[
y^2 + 12y + 36 = 5x - 16 + 36 = 5x + 20
\]

\[
(y + 6)^2 = 5(x + 4)
\]

The parabola opens to the right. It has vertex $V(-4, -6)$. From $4c = 5$, we get $c = \frac{5}{4} = 1.25$. The focus is $c = 1.25$ units to the right of $V$: $F(-2.75, -6)$. The (vertical) directrix is $c = 1.25$ units to the left of $V$: $x = -5.25$. The (horizontal) axis is through $V$: $y = -6$.

(2) We complete the square on $x$, and move $y$ to the other side.

\[
5x^2 + 30x = -24y + 51
\]

\[
5(x^2 + 6x + 9) = -24y + 51 + 5(9)
\]

\[
5(x + 3)^2 = -24y + 96 = -24(y - 4)
\]

\[
(x + 3)^2 = -\frac{24}{5}(y - 4)
\]

In the last line, we divided by 5 for the squared part not to have any coefficient. The parabola opens downward. It has vertex $V(-3, 4)$.

From $4c = \frac{24}{5}$, we get $c = \frac{6}{5} = 1.2$. The focus is $c = 1.2$ units below $V$: $F(-3, 2.8)$. The (horizontal) directrix is $c = 1.2$ units above $V$: $y = 5.2$. The (vertical) axis is through $V$: $x = -3$. 

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**Example 1.2.5.** A parabola has focus $F(7, 9)$ and directrix $y = 3$. Find its standard equation.

**Solution.** The directrix is horizontal, and the focus is above it. The parabola then opens upward and its standard equation has the form $(x - h)^2 = 4c(y - k)$. Since the distance from the focus to the directrix is $2c = 9 - 3 = 6$, then $c = 3$. Thus, the vertex is $V(7, 6)$, the point 3 units below $F$. The standard equation is then $(x - 7)^2 = 12(y - 6)$.

**Seatwork/Homework 1.2.2**

1. Determine the vertex, focus, directrix, and axis of symmetry of the parabola with equation $x^2 - 6x + 5y = -34$. Sketch the graph, and include these points and lines.

   Answer: vertex $(3, -5)$, focus $(3, -6.25)$, directrix $y = -3.75$, axis $x = 3$
2. A parabola has focus \( F(-2, -5) \) and directrix \( x = 6 \). Find the standard equation of the parabola.

Answer: \((y + 5)^2 = -16(x - 2)\)

1.2.3. Situational Problems Involving Parabolas

We now solve some situational problems involving parabolas.

Example 1.2.6. A satellite dish has a shape called a paraboloid, where each cross-section is a parabola. Since radio signals (parallel to the axis) will bounce off the surface of the dish to the focus, the receiver should be placed at the focus. How far should the receiver be from the vertex, if the dish is 12 ft across, and 4.5 ft deep at the vertex?

Solution. The second figure above shows a cross-section of the satellite dish drawn on a rectangular coordinate system, with the vertex at the origin. From the problem, we deduce that \((6, 4.5)\) is a point on the parabola. We need the distance of the focus from the vertex, i.e., the value of \(c\) in \(x^2 = 4cy\).

\[
\begin{align*}
\frac{x^2}{4c} &= y \\
\frac{6^2}{4c} &= 4.5 \\
6^2 &= 4c(4.5) \\
c &= \frac{6^2}{4 \cdot 4.5} = 2
\end{align*}
\]

Thus, the receiver should be 2 ft away from the vertex.

Example 1.2.7. The cable of a suspension bridge hangs in the shape of a parabola. The towers supporting the cable are 400 ft apart and 150 ft high. If the cable, at its lowest, is 30 ft above the bridge at its midpoint, how high is the cable 50 ft away (horizontally) from either tower?
Solution. Refer to the figure above, where the parabolic cable is drawn with its vertex on the y-axis 30 ft above the origin. We may write its equation as \((x - 0)^2 = a(y - 30)\); since we don’t need the focal distance, we use the simpler variable \(a\) in place of \(4c\). Since the towers are 150 ft high and 400 ft apart, we deduce from the figure that \((200, 150)\) is a point on the parabola.

\[
x^2 = a(y - 30)
\]

\[
200^2 = a(150 - 30)
\]

\[
a = \frac{200^2}{120} = \frac{10000}{3}
\]

The parabola has equation \(x^2 = \frac{10000}{3}(y - 30)\), or equivalently, \(y = 0.003x^2 + 30\). For the two points on the parabola 50 ft away from the towers, \(x = 150\) or \(x = -150\). If \(x = 150\), then

\[
y = 0.003(150^2) + 30 = 97.5.
\]

Thus, the cable is 97.5 ft high 50 ft away from either tower. (As expected, we get the same answer from \(x = -150\).)

Seatwork/Homework 1.2.3

*1. A satellite dish in the shape of a paraboloid is 10 ft across, and 4 ft deep at its vertex. How far is the receiver from the vertex, if it is placed at the focus? Round off your answer to 2 decimal places. (Refer to Example 1.2.6.)

Answer: 1.56 ft

Exercises 1.2

1. Determine the vertex, focus, directrix, and axis of symmetry of the parabola with the given equation. Sketch the graph, and include these points and lines.
(a) \( x^2 = -4y \)  
(b) \( 3y^2 = 24x \)  
(c) \( \left( y + \frac{5}{2} \right)^2 = -5 \left( x - \frac{9}{2} \right) \)  
(d) \( x^2 + 6x + 8y = 7 \)  
(e) \( y^2 - 12x + 8y = -40 \)  
(f) \( 16x^2 + 72x - 112y = -221 \)

Answer:

<table>
<thead>
<tr>
<th>Item</th>
<th>Vertex</th>
<th>Focus</th>
<th>Directrix</th>
<th>Axis of Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(0,0)</td>
<td>(0,-1)</td>
<td>( y = 1 )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>(b)</td>
<td>(0,0)</td>
<td>(2,0)</td>
<td>( x = -2 )</td>
<td>( y = 0 )</td>
</tr>
<tr>
<td>(c)</td>
<td>(4.5,-2.5)</td>
<td>(3.25,-2.5)</td>
<td>( x = 5.75 )</td>
<td>( y = -2.5 )</td>
</tr>
<tr>
<td>(d)</td>
<td>(-3,2)</td>
<td>(-3,0)</td>
<td>( y = 4 )</td>
<td>( x = -3 )</td>
</tr>
<tr>
<td>(e)</td>
<td>(2,-4)</td>
<td>(5,-4)</td>
<td>( x = -1 )</td>
<td>( y = -4 )</td>
</tr>
<tr>
<td>(f)</td>
<td>(-2.25,1.25)</td>
<td>(-2.25,3)</td>
<td>( y = -0.5 )</td>
<td>( x = -2.25 )</td>
</tr>
</tbody>
</table>

2. Find the standard equation of the parabola which satisfies the given conditions. **Teaching Notes**

   It is helpful to draw a diagram for each item.

   (a) vertex (1,-9), focus (-3,-9)  
   Answer: \( (y+9)^2 = -16(x-1) \)
(b) vertex \((-8, 3)\), directrix \(x = -10.5\) \[\text{Answer: } (y - 3)^2 = 10(x + 8)\]

(c) vertex \((-4, 2)\), focus \((-4, -1)\) \[\text{Answer: } (x + 4)^2 = -12(y - 2)\]

(d) focus \((7, 11)\), directrix \(x = 1\) \[\text{Answer: } (y - 11)^2 = 12(x - 4)\]

(e) focus \((7, 11)\), directrix \(y = 4\) \[\text{Answer: } (x - 7)^2 = 14(y - 7.5)\]

(f) vertex \((-5, -7)\), vertical axis of symmetry, through the point \(P(7, 11)\) \[\text{Answer: } (x + 5)^2 = 8(y + 7)\]

\[\text{Solution.}\] Since the axis is vertical and \(P\) is above the vertex, then the parabola opens upward and has equation of the form \((x + 5)^2 = 4c(y + 7)\). We plug the coordinates of \(P\): \((7 + 5)^2 = 4c(11 + 7)\). We then get \(c = 2\). Thus, we have \((x + 5)^2 = 8(y + 7)\).

(g) vertex \((-5, -7)\), horizontal axis of symmetry, through the point \(P(7, 11)\) \[\text{Answer: } (y + 7)^2 = 27(x + 5)\]

\[\text{Solution.}\] Since the axis is horizontal and \(P\) is to the right of the vertex, then the parabola opens to the right and has equation of the form \((y + 7)^2 = 4c(x + 5)\). We plug the coordinates of \(P\): \((11 + 7)^2 = 4c(7 + 5)\). We then get \(c = 6.75\). Thus, we have \((y + 7)^2 = 27(x + 5)\).

3. A satellite dish shaped like a paraboloid, has diameter 2.4 ft and depth 0.9 ft. If the receiver is placed at the focus, how far should the receiver be from the vertex? \[\text{Answer: } 0.4 \text{ ft}\]

4. If the diameter of the satellite dish from the previous problem is doubled, with the depth kept the same, how far should the receiver be from the vertex? \[\text{Answer: } 1.6 \text{ ft}\]

*5. A satellite dish is shaped like a paraboloid, with the receiver placed at the focus. It is to have a depth of 0.44 m at the vertex, with the receiver placed 0.11 m away from the vertex. What should the diameter of the satellite dish be? \[\text{Answer: } 0.88 \text{ m}\]

*6. A flashlight is shaped like a paraboloid, so that if its light bulb is placed at the focus, the light rays from the bulb will then bounce off the surface in a focused direction that is parallel to the axis. If the paraboloid has a depth of 1.8 in and the diameter on its surface is 6 in, how far should the light source be placed from the vertex? \[\text{Answer: } 1.25 \text{ in}\]

7. The towers supporting the cable of a suspension bridge are 1200 m apart and 170 m above the bridge it supports. Suppose the cable hangs, following the shape of a parabola, with its lowest point 20 m above the bridge. How high is the cable 120 m away from a tower? \[\text{Answer: } 116 \text{ m}\]
Lesson 1.3. Ellipses

Time Frame: 3 one-hour sessions

Learning Outcomes of the Lesson

At the end of the lesson, the student is able to:

(1) define an ellipse;
(2) determine the standard form of equation of an ellipse;
(3) graph an ellipse in a rectangular coordinate system; and
(4) solve situational problems involving conic sections (ellipses).

Lesson Outline

(1) Definition of an ellipse
(2) Derivation of the standard equation of an ellipse
(3) Graphing ellipses
(4) Solving situational problems involving ellipses

Introduction

An ellipse is one of the conic sections that most students have not encountered formally before, unlike circles and parabolas. Its shape is a bounded curve which looks like a flattened circle. The orbits of the planets in our solar system around the sun happen to be elliptical in shape. Also, just like parabolas, ellipses have reflective properties that have been used in the construction of certain structures (shown in some of the practice problems). We will see some properties and applications of ellipses in this section.

1.3.1. Definition and Equation of an Ellipse

Consider the points \( F_1(-3, 0) \) and \( F_2(3, 0) \), as shown in Figure 1.19. What is the sum of the distances of \( A(4,2.4) \) from \( F_1 \) and from \( F_2 \)? How about the sum of the distances of \( B \) (and \( C(0,-4) \)) from \( F_1 \) and from \( F_2 \)?

\[
AF_1 + AF_2 = 7.4 + 2.6 = 10
\]
\[
BF_1 + BF_2 = 3.8 + 6.2 = 10
\]
\[
CF_1 + CF_2 = 5 + 5 = 10
\]

There are other points \( P \) such that \( PF_1 + PF_2 = 10 \). The collection of all such points forms a shape called an *ellipse.*
Let $F_1$ and $F_2$ be two distinct points. The set of all points $P$, whose distances from $F_1$ and from $F_2$ add up to a certain constant, is called an ellipse. The points $F_1$ and $F_2$ are called the foci of the ellipse.

Given are two points on the $x$-axis, $F_1(-c,0)$ and $F_2(c,0)$, the foci, both $c$ units away from their center $(0,0)$. See Figure 1.20. Let $P(x,y)$ be a point on the ellipse. Let the common sum of the distances be $2a$ (the coefficient 2 will make computations simpler). Thus, we have $PF_1 + PF_2 = 2a$.

\[
PF_1 = 2a - PF_2 \\
\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2} \\
x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 \\
a\sqrt{(x-c)^2 + y^2} = a^2 - cx \\
a^2[x^2 - 2cx + c^2 + y^2] = a^4 - 2a^2cx + c^2x^2 \\
(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2 = a^2(a^2 - c^2) \\
b^2x^2 + a^2y^2 = a^2b^2 \quad \text{by letting } b = \sqrt{a^2 - c^2}, \text{ so } a > b \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

When we let $b = \sqrt{a^2 - c^2}$, we assumed $a > c$. To see why this is true, look at $\triangle PF_1F_2$ in Figure 1.20. By the Triangle Inequality, $PF_1 + PF_2 > F_1F_2$, which implies $2a > 2c$, so $a > c$.

We collect here the features of the graph of an ellipse with standard equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a > b. \text{ Let } c = \sqrt{a^2 - b^2}.
\]
(1) *center*: origin $\{0, 0\}$

(2) *foci*: $F_1(-c, 0)$ and $F_2(c, 0)$

- Each focus is $c$ units away from the center.
- For any point on the ellipse, the sum of its distances from the foci is $2a$.

(3) *vertices*: $V_1(-a, 0)$ and $V_2(a, 0)$

- The vertices are points on the ellipse, collinear with the center and foci.
- If $y = 0$, then $x = \pm a$. Each vertex is $a$ units away from the center.
- The segment $V_1V_2$ is called the major axis. Its length is $2a$. It divides the ellipse into two congruent parts.

(4) *covertices*: $W_1(0, -b)$ and $W_2(0, b)$

- The segment through the center, perpendicular to the major axis, is the minor axis. It meets the ellipse at the covertices. It divides the ellipse into two congruent parts.
- If $x = 0$, then $y = \pm b$. Each covertex is $b$ units away from the center.
- The minor axis $W_1W_2$ is $2b$ units long. Since $a > b$, the major axis is longer than the minor axis.

**Example 1.3.1.** Give the coordinates of the foci, vertices, and covertices of the ellipse with equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$ 

Sketch the graph, and include these points.

**Solution.** With $a^2 = 25$ and $b^2 = 9$, we have $a = 5$, $b = 3$, and $c = \sqrt{a^2 - b^2} = 4$.

**foci**: $F_1(-4, 0), F_2(4, 0)$

**vertices**: $V_1(-5, 0), V_2(5, 0)$

**covertices**: $W_1(0, -3), W_2(0, 3)$
Example 1.3.2. Find the (standard) equation of the ellipse whose foci are $F_1(-3,0)$ and $F_2(3,0)$, such that for any point on it, the sum of its distances from the foci is 10. See Figure 1.19.

Solution. We have $2a = 10$ and $c = 3$, so $a = 5$ and $b = \sqrt{a^2 - c^2} = 4$. The equation is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$ 

\square

Seatwork/Homework 1.3.1

1. Give the coordinates of the foci, vertices, and covertices of the ellipse with equation $\frac{x^2}{169} + \frac{y^2}{25} = 1$. Sketch the graph, and include these points.

   Answer: foci: $F_1(-12,0)$ and $F_2(12,0)$, vertices: $V_1(-13,0)$ and $V_2(13,0)$, covertices: $W_1(0,-5)$ and $W_2(0,5)$

2. Find the equation in standard form of the ellipse whose foci are $F_1(-8,0)$ and $F_2(8,0)$, such that for any point on it, the sum of its distances from the foci is 20.

   Answer: $\frac{x^2}{100} + \frac{y^2}{36} = 1$
1.3.2. More Properties of Ellipses

Some ellipses have their foci aligned vertically, and some have centers not at the origin. Their standard equations and properties are given in the box. The derivations are more involved, but are similar to the one above, and so are not shown anymore.

<table>
<thead>
<tr>
<th>Center</th>
<th>Corresponding Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>![Graph of ellipse with center at (0, 0) and foci F1 and F2]</td>
</tr>
<tr>
<td>(x, y)</td>
<td>![Graph of ellipse with center at (h, k) and foci F1 and F2]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Center</th>
<th>Major Axis</th>
<th>Minor Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>Horizontal</td>
<td>Vertical</td>
</tr>
<tr>
<td>(h, k)</td>
<td>Vertical</td>
<td>Horizontal</td>
</tr>
</tbody>
</table>

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

\[
\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1
\]

\[
\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1
\]

\[
\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1
\]
In all four cases above, \( a > b \) and \( c = \sqrt{a^2 - b^2} \). The foci \( F_1 \) and \( F_2 \) are \( c \) units away from the center. The vertices \( V_1 \) and \( V_2 \) are \( a \) units away from the center, the major axis has length \( 2a \), the covertices \( W_1 \) and \( W_2 \) are \( b \) units away from the center, and the minor axis has length \( 2b \). Recall that, for any point on the ellipse, the sum of its distances from the foci is \( 2a \).

In the standard equation, if the \( x \)-part has the bigger denominator, the ellipse is horizontal. If the \( y \)-part has the bigger denominator, the ellipse is vertical.

**Example 1.3.3.** Give the coordinates of the center, foci, vertices, and covertices of the ellipse with the given equation. Sketch the graph, and include these points.

1. \( \frac{(x + 3)^2}{24} + \frac{(y - 5)^2}{49} = 1 \)
2. \( 9x^2 + 16y^2 - 126x + 64y = 71 \)

**Solution.** (1) From \( a^2 = 49 \) and \( b^2 = 24 \), we have \( a = 7 \), \( b = 2\sqrt{6} \approx 4.9 \), and \( c = \sqrt{a^2 - b^2} = 5 \). The ellipse is vertical.

- **center:** \((-3, 5)\)
- **foci:** \( F_1(-3, 0), F_2(-3, 10) \)
- **vertices:** \( V_1(-3, -2), V_2(-3, 12) \)
- **covertices:** \( W_1(-3 - 2\sqrt{6}, 5) \approx (-7.9, 5) \)
  \( W_2(-3 + 2\sqrt{6}, 5) \approx (1.9, 5) \)
We first change the given equation to standard form.

\[
9(x^2 - 14x) + 16(y^2 + 4y) = 71
\]

\[
9(x^2 - 14x + 49) + 16(y^2 + 4y + 4) = 71 + 9(49) + 16(4)
\]

\[
9(x - 7)^2 + 16(y + 2)^2 = 576
\]

\[
\frac{(x - 7)^2}{64} + \frac{(y + 2)^2}{36} = 1
\]

We have \(a = 8\) and \(b = 6\). Thus, \(c = \sqrt{a^2 - b^2} = 2\sqrt{7} \approx 5.3\). The ellipse is horizontal.

center: \( (7, -2) \)

foci: \( F_1(7 - 2\sqrt{7}, -2) \approx (1.7, -2) \)

\( F_2(7 + 2\sqrt{7}, -2) \approx (12.3, -2) \)

vertices: \( V_1(-1, -2), V_2(15, -2) \)

covertices: \( W_1(7, -8), W_2(7, 4) \)

Example 1.3.4. The foci of an ellipse are \((-3, -6)\) and \((-3, 2)\). For any point on the ellipse, the sum of its distances from the foci is 14. Find the standard equation of the ellipse.
Solution. The midpoint $(-3, -2)$ of the foci is the center of the ellipse. The ellipse is vertical (because the foci are vertically aligned) and $c = 4$. From the given sum, $2a = 14$ so $a = 7$. Also, $b = \sqrt{a^2 - c^2} = \sqrt{33}$. The equation is $\frac{(x + 3)^2}{33} + \frac{(y + 2)^2}{49} = 1$. □

Example 1.3.5. An ellipse has vertices $(2 - \sqrt{61}, -5)$ and $(2 + \sqrt{61}, -5)$, and its minor axis is 12 units long. Find its standard equation and its foci.

Solution. The midpoint $(2, -5)$ of the vertices is the center of the ellipse, which is horizontal. Each vertex is $a = \sqrt{61}$ units away from the center. From the length of the minor axis, $2b = 12$ so $b = 6$. The standard equation is $\frac{(x - 2)^2}{61} + \frac{(y + 5)^2}{36} = 1$. Each focus is $c = \sqrt{a^2 - b^2} = 5$ units away from $(2, -5)$, so their coordinates are $(-3, -5)$ and $(7, -5)$.

□

Seatwork/Homework 1.3.2

1. Give the coordinates of the center, foci, vertices, and covertices of the ellipse with equation $41x^2 + 16y^2 + 246x - 192y + 289 = 0$. Sketch the graph, and include these points.

Answer: center $C(-3, 6)$, foci $F_1(-3, 1)$ and $F_2(-3, 11)$, vertices $V_1(-3, 6 - \sqrt{41})$ and $V_2(-3, 6 + \sqrt{41})$, covertices $W_1(-7, 6)$ and $W_2(1, 6)$
2. An ellipse has vertices \((-10, -4)\) and \((6, -4)\), and covertices \((-2, -9)\) and \((-2, 1)\). Find its standard equation and its foci.

Answer: \[
\frac{(x + 2)^2}{64} + \frac{(y + 4)^2}{25} = 1, \quad \text{foci } (-2 - \sqrt{39}, -4) \text{ and } (-2 + \sqrt{39}, -4)
\]

1.3.3. Situational Problems Involving Ellipses

We now apply the concept of ellipse to some situational problems.

*Example 1.3.6. A tunnel has the shape of a semiellipse that is 15 ft high at the center, and 36 ft across at the base. At most how high should a passing truck be, if it is 12 ft wide, for it to be able to fit through the tunnel? Round off your answer to two decimal places.

Solution. Refer to the figure above. If we draw the semiellipse on a rectangular coordinate system, with its center at the origin, an equation of the ellipse which contains it, is

\[
\frac{x^2}{18^2} + \frac{y^2}{15^2} = 1.
\]

To maximize its height, the corners of the truck, as shown in the figure, would have to just touch the ellipse. Since the truck is 12 ft wide, let the point \((6, n)\) be the corner of the truck in the first quadrant, where \(n > 0\), is the (maximum) height of the truck. Since this point is on the ellipse, it should fit the equation. Thus, we have

\[
\frac{6^2}{18^2} + \frac{n^2}{15^2} = 1
\]

\[
\frac{6^2}{18^2} = \frac{n^2}{15^2} = 1 - \frac{6^2}{18^2}
\]

\[
n^2 = 15^2 \left(1 - \frac{6^2}{18^2}\right)
\]

\[
n = 10 \sqrt{2} \approx 14.14 \text{ ft}
\]

\[
\square
\]

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